Brief Contributions

A Geometric Theorem for Network Design

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Abstract—Consider an infinite square grid G. How many discs of given radius r, centered at the vertices of G, are required, in the worst case, to completely cover an arbitrary disc of radius r placed on the plane? We show that this number is an integer in the set $\{3, 4, 5, 6\}$ whose value depends on the ratio of r to the grid spacing. One application of this result is to design facility location algorithms with constant approximation factors. Another application is to determine if a grid network design, where facilities are placed on a regular grid in a way that each potential customer is within a reasonably small radius around the facility, is cost effective in comparison to a nongrid design. This can be relevant to determine a cost effective design for base station placement in a wireless network.

Index Terms—Facility location, combinatorial geometry, network design, wireless networks.

1 Introduction

1.1 Problem Formulation

In this paper, we answer the following basic combinatorial geometry question: How many discs of given radius r, centered at the vertices of an infinite square grid, are required, in the worst case, to completely cover an arbitrary disc of radius r placed on the plane and how is this worst case achieved?

The answer depends on the ratio of the radius r to the grid spacing L, as illustrated in Fig. 1. On the left-hand side of the figure six solid line discs, centered at grid vertices, are required to cover a dashed line disc placed halfway between two adjacent grid vertices. If the ratio of r to the grid spacing L is increased (right-hand side of the figure), then fewer discs centered at grid vertices are required to cover the same dashed line disc. The following geometric theorem gives a complete solution.

Theorem 1. *Consider a square lattice where the distance between two neighboring lattice vertices is L. Call a disc of fixed radius r, centered at a lattice vertex, a grid disc. The number $\mathcal{N}$ of grid discs that are necessary and sufficient to cover any disc of radius r placed on the plane, is given by:*

- *CASE 1. For $r/L < \frac{\sqrt{2}}{2}$, $\mathcal{N}$ does not exist.*
- *CASE 2. For $\frac{\sqrt{2}}{2} \leq r/L < \frac{\sqrt{10}}{4}$, $\mathcal{N} = 6$.*
- *CASE 3. For $\frac{\sqrt{10}}{4} \leq r/L < 1$, $\mathcal{N} = 5$.*
- *CASE 4. For $1 \leq r/L < \frac{5\sqrt{2}}{4}$, $\mathcal{N} = 4$.*
- *CASE 5. For $r/L \geq \frac{5\sqrt{2}}{4}$, $\mathcal{N} = 3$.*

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1.2 Applications

One application of our geometric result is in the design of facility location algorithms. Consider the problem of locating facilities such that all potential customers will be within a reasonably small radius around the facility. The problem can be abstracted as a disc covering problem: Given points in the plane, identify a minimally sized set of discs of given radius that cover all the points. This is known in the computer science literature as the *geometric disc covering problem* and it is NP-complete [1]. Polynomial approximation algorithms that provide a suboptimal solution that is within a constant approximation factor to the optimal one have been proposed in [2], [4], [7]. These algorithms are based on the observation that the number of possible discs positions can be bounded by assuming (without loss of generality) that any disc that covers at least two points has two of these points on its border. An exhaustive search on all possible disc positions leads to an optimal solution, but runs in a time that is exponential in the number n of points to cover. By performing a search on a subset of the possible discs positions, the running time of the algorithm becomes polynomial in the number n of points to cover and the solution is still guaranteed to be within a constant factor from the optimal one.

An alternative approach could be to limit the number of possible disc positions a priori to be on a regular grid and to design algorithms that perform a search on this fixed number of grid vertices that is independent of n. Clearly, if n is large and the grid is relatively sparse, i.e., the number of grid vertices is small, such algorithms can be computationally more efficient and still exhibit constant worst-case approximation factors, which are provided by our geometric theorem. A simple demonstration of this is given by analyzing the performance of the algorithm in [7] on the grid as is detailed in the technical report [3]. This report also contains a detailed comparison with other approximation algorithms that appeared in the literature, emphasizing the trade off between the quality of the approximation and the running time of the algorithm. It is shown that the running time of grid algorithms compares favorably with existing solutions, while maintaining constant approximation factors.

Another application is in the evaluation of a grid network design compared to a nongrid design. Consider, for example, facilities to be radio base stations that must cover a set of potential customers. The problem is where to position base stations, according to a distribution of demand points, to achieve optimum quality of service, with minimum costs. In this case, the number of grid stations required to cover the same demand points that are covered by a single nongrid station is given, in the worst case, by Theorem 1. Assuming that manufacturing on the grid leads to a lower cost per base station, the theorem can then be used to decide whether a grid network design is cost effective. A simple way to do this is to substitute, for each nongrid disc, the required number of grid discs that fully cover it and compute the new cost accordingly.

1.3 A Critique of Facility Location

We point out that facility location problems such as those that represent our motivating application are well-studied problems in computer science, but they also represent an abstraction and, hence, a simplification of real problems. In reality, facilities cannot be placed anywhere on the plane, but their potential location can be influenced by a variety of physical and economic considerations. Similarly, only some cities (or part of cities) resemble a regular grid pattern that can be exploited by the location of the facilities. Moreover, having a *fixed* coverage radius and knowing the location of the demand points might not always be possible.

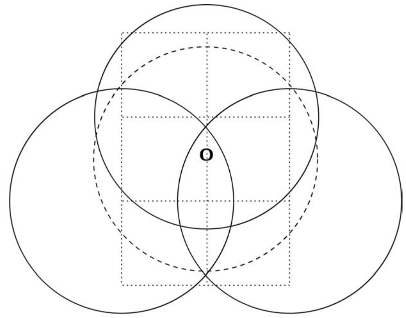


Fig. 1. Scaling the picture. On the left-hand side of the figure, the dashed line disc requires six discs centered at lattice vertices to be fully covered. On the right-hand side of the figure, it requires only three discs.

There is no doubt that the abstraction can be improved, relaxing some or all previous assumptions. The power of the abstraction is, however, in its analytical solution which can be used as a guideline at the early design stage of a network. We point out that similar simplistic assumptions have often been used in the literature to formally derive properties of networks. For example, a circle (or a hexagon to approximate the circle's shape) that bounds the maximal range of a facility is universally adopted as a first-order approximation at the early design stage of a wireless cellular network [10] and has been assumed in the calculation of the throughput capacity of ad hoc wireless networks [5], [6], [9]. The concept of identifying demand points and using them to model the expected network traffic has recently been standardized by the the International Telecommunication Union [8], [11]. According to their definition, a demand node represents the center of an area with a certain traffic demand and each node stands for the same portion of traffic load. Hence, different traffic patterns correspond to different node distributions: Highly populated business districts typically lead to dense distributions of demand nodes, while suburban and rural areas lead to sparser distributions. Finally, the assumption of having a city formed by regularly spaced blocks applies well to some US suburban and urban areas.

2 Nothing But Proofs

This long section is devoted to the proof of Theorem 1. Conclusions and future work are discussed in the next section. The proofs below make use of simple geometric arguments, but are by no means trivial. They involve finding a disc that requires the prescribed number of grid discs to be fully covered and then finding different configurations of grid discs that are sufficient to cover any disc on the plane. The number of these configurations that we need to find increases with r/L.

To simplify the notation, in the following, we fix $L = 1$.

2.1 Proof of CASE 1

For $r < \frac{\sqrt{2}}{2}$, the grid discs do not cover the plane compactly since they do not cover the centers of the lattice squares. It follows that any number of grid discs is not sufficient to cover any disc that covers the center of a lattice square.

2.2 Proof of CASE 2

The necessary condition is proven by showing that there exists a disc that requires six grid discs to be covered. The sufficient condition is proven using a tiling argument: We first show that there exists a triangle ABC, such that any disc centered inside triangle ABC is covered by six grid discs and then we show that, by symmetry, we can tile the entire plane using triangles that have this property.

2.2.1 Necessary Condition

Consider $r \geq \frac{\sqrt{2}}{2}$ and a disc centered halfway between two neighboring lattice vertices. Such a disc is the dashed disc depicted in Fig. 2. We have that the six (solid) grid discs depicted in Fig. 2 are necessary to completely cover the dashed disc if $\overline{BC} > r$, i.e., when the four shaded areas in Fig. 2 are not null. By repeatedly applying the Pythagorean theorem, we have:

$$\overline{BC} = \sqrt{\left(1 - \sqrt{r^2 - \frac{1}{16}}\right)^2 + \frac{9}{16}}; \tag{1}$$

imposing $\overline{BC} > r$, we obtain $r < \frac{\sqrt{10}}{4}$.

2.2.2 Sufficient Condition

Call $\mathcal{A}$ the area covered by the six grid discs in Fig. 3. Any point inside triangle ABC has distance greater than r from the border of area $\mathcal{A}$. Therefore, a disc can be centered inside triangle ABC and be covered by the six grid discs depicted in Fig. 3. By symmetry, we can tile the plane with triangles inside which discs can be centered and covered by six grid discs.

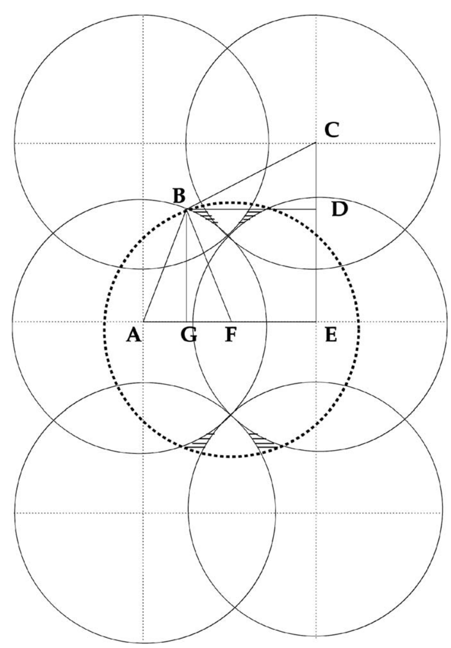


Fig. 2. CASE 2, necessary condition. The distance between the two lattice vertices A and E is 1. Therefore, $\overline{AF} = \overline{FE} = \frac{1}{2}$. Since $\overline{AB} = \overline{BF} = r$, $\overline{AG} = \overline{GF} = \frac{1}{4}$.

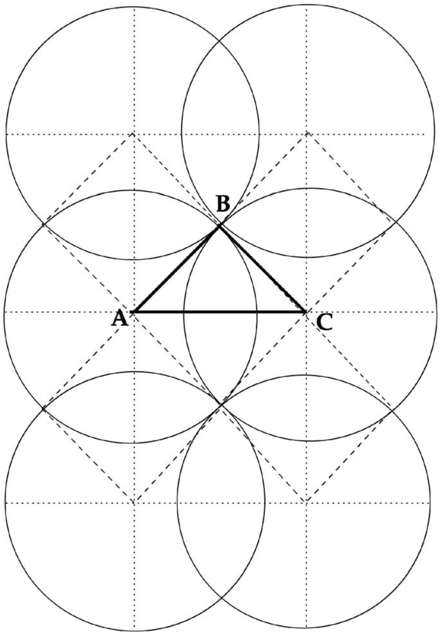


Fig. 3. CASE 2, sufficient condition. Any disc centered inside triangle ABC is covred by the six grid discs.

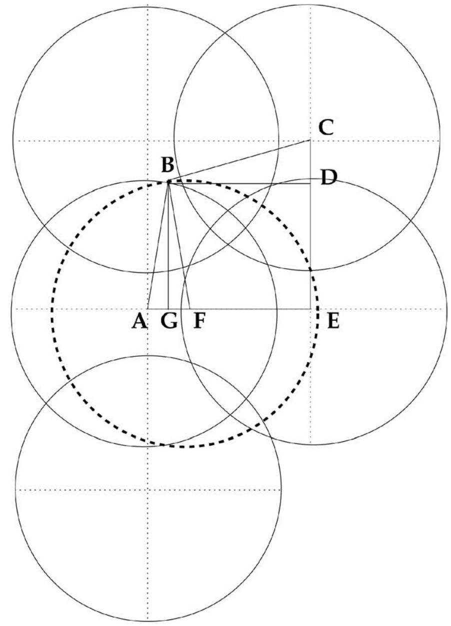


Fig. 4. CASE 3, necessary condition. Point F is the center of the dashed disc and is shifted by ϵ to the left from halfway between the lattice vertices A and E. Therefore, we have: $\overline{AF} = \frac{1}{2} - \epsilon$, $\overline{FE} = \frac{1}{2} + \epsilon$.

2.3 Proof of CASE 3

2.3.1 Necessary Condition

Consider $r \geq \frac{\sqrt{10}}{4}$ and a disc centered at a distance ϵ to the left from halfway between two neighboring lattice vertices (dashed disc in Fig. 4). By the same reasoning of CASE 2, we have that the five (solid) grid discs depicted in Fig. 4 are necessary to completely cover the dashed disc, if $\overline{BC} > r$. By repeatedly applying the Pythagorean theorem, we have, in this case:

$$\overline{BC} = \sqrt{\left[1 - \sqrt{r^2 - \left(\frac{1}{4} - \frac{\epsilon}{2}\right)^2}\right]^2 + \left(\frac{3}{4} + \frac{\epsilon}{2}\right)^2}; \tag{2}$$

imposing $\overline{BC} > r$, we obtain:

$$\sqrt{\frac{\epsilon^2 + \epsilon}{2} + \frac{5}{8}} > r. \tag{3}$$

By symmetry, we can restrict the ϵ range: $0 < \epsilon < \frac{1}{2}$ and, for any $r < 1$, inequality (3) is verified.

2.3.2 Sufficient Condition

Call $\mathcal{A}$ the area covered by the five grid discs in Fig. 5. Any point inside triangle AOB has distance greater than r from the border of area $\mathcal{A}$. Therefore, a disc can be centered inside triangle AOB and be covered by the five grid discs depicted in Fig. 5. By symmetry, we can tile the plane with triangles inside which discs can be centered and covered by five grid discs.

2.4 Proof of CASE 4

2.4.1 Necessary Condition

Consider $r \geq 1$ and a disc placed at the center of a lattice square. If we place the grid discs as depicted in Fig. 6b, four grid discs are always necessary to cover the dashed disc placed at the center of a lattice square because:

$$\overline{EF} = \overline{DF} > \overline{OF} = r; \tag{4}$$

the same holds whenever two grid discs are centered at neighboring lattice vertices or at lattice vertices on the same diagonal of a lattice square. If we examine the remaining possible placement of grid discs, depicted in Fig. 6a, we have that four grid discs are necessary only if $\overline{BC} = \overline{CD} > r$. By the Pythagorean theorem, we have, in this case:

$$\overline{BC} = \sqrt{\left(r - \frac{\sqrt{2}}{2}\right)^2 + 2}; \tag{5}$$

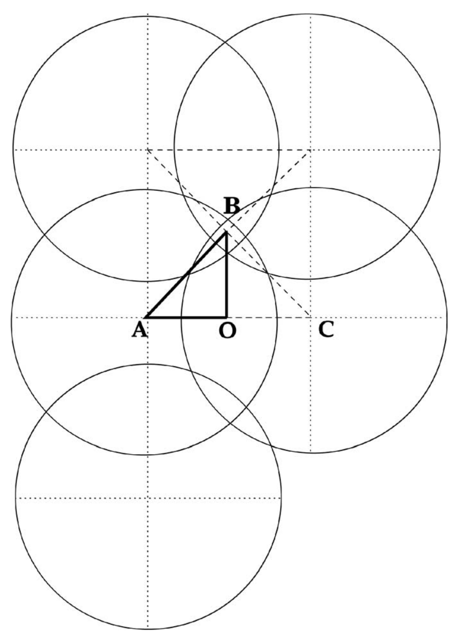
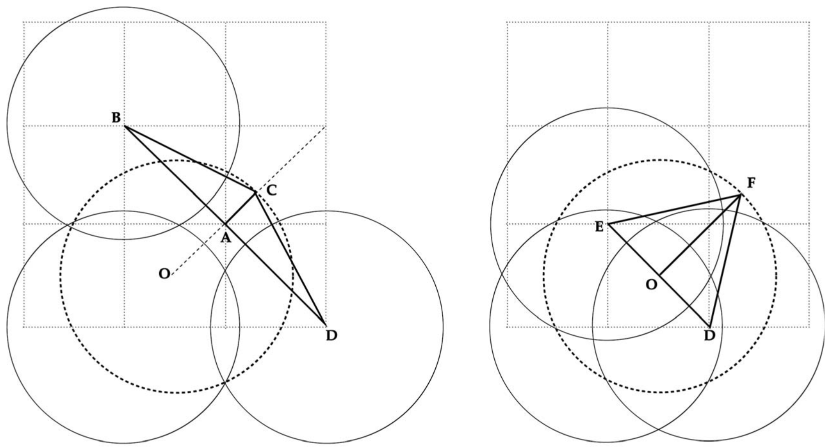
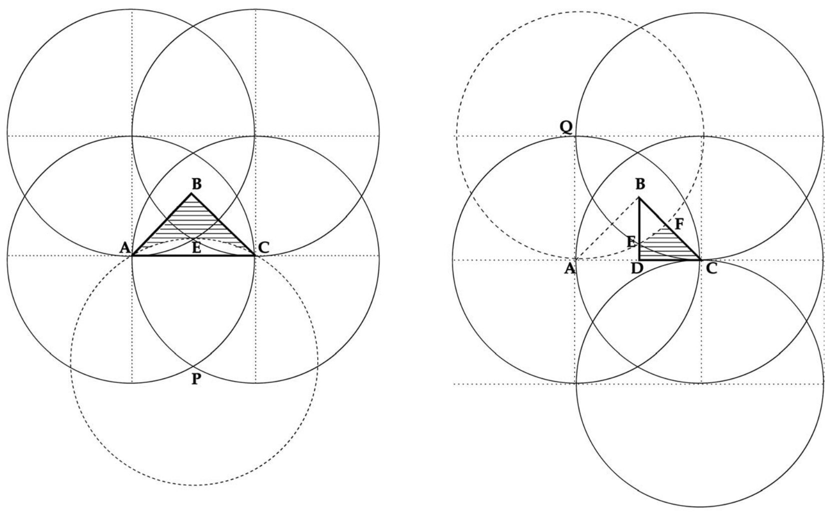


Fig. 5. CASE 3, sufficient condition. Point O is halfway between lattice vertices A and C. Any disc centered inside triangle AOB is covered by the five grid discs.



(a) (b)

Fig. 6. CASE 4, necessary condition. Point O is at the center of a lattice square. (a) The dashed disc centered at point O requires one more grid disc to be covered if $\overline{BC} > r$. (b) The dashed disc centered at point O always requires one more grid disc to be covered.



(a) (b)

Fig. 7. CASE 4, sufficient condition. Four grid discs cover any disc centered inside the shaded areas. Intersecting the two shaded areas, we obtain triangle BCD.

imposing $\overline{BC} > r$ we obtain $r < 5\frac{\sqrt{2}}{4}$.

2.4.2 Sufficient Condition

Consider the left section of Fig. 7. The four grid discs cover any disc centered inside the shaded area $ABCE$. This area is defined by the triangle ABC and by the circle centered at point P. This circle is the locus of the centers of the discs to be covered that touch point P. Any disc centered inside the remaining area ACE is not covered by the four grid discs. Consider now Fig. 7b, the four (solid) grid discs cover, in this case, any disc centered inside the shaded area $CDEF$. This area is defined by triangle BCD and by the circle centered at point Q. This circle is the locus of the centers of the discs to be covered that touch point Q. Such a circle passes by point E, therefore, adjoining the two shaded areas $ABCE$ and $CDEF$, we fully cover the area of triangle BCD. Any disc centered inside triangle BCD is covered by four grid discs: the four depicted in Fig. 7a or the four depicted in Fig. 7b. By symmetry, the same holds for triangle ABD and we can tile the plane with triangles inside which discs can be centered and covered by four grid discs.

2.5 Proof of CASE 5

2.5.1 Necessary Condition

By symmetry, any two discs with the same radius, centered far apart by an arbitrary $\epsilon > 0$, cover less than half of each other's perimeter; therefore, any grid disc must cover less than half of the perimeter of any other disc not centered at a lattice point. It follows that any two grid discs must cover less than the entire perimeter of any other disc not centered at a lattice point. Hence, any disc not centered at a lattice point requires at least three grid discs to be covered.

2.5.2 Sufficient Condition

It is enough to prove this condition when r is minimum, therefore, we carry out calculations fixing $r = \frac{5\sqrt{2}}{4}$. We consider three different placements of three grid discs on the lattice and show that they are enough to cover any disc arbitrarily placed on the plane. Consider Fig. 8a. Taking point O as the origin of the coordinate system, the three grid discs are placed at points: $(-1,0), (-1,2), (1,0)$. The coordinates of point P are calculated by applying the Pythagorean theorem to triangle AOP obtaining: $P = (0, -\frac{\sqrt{34}}{4})$. Therefore, the locus of the centers of the discs to be

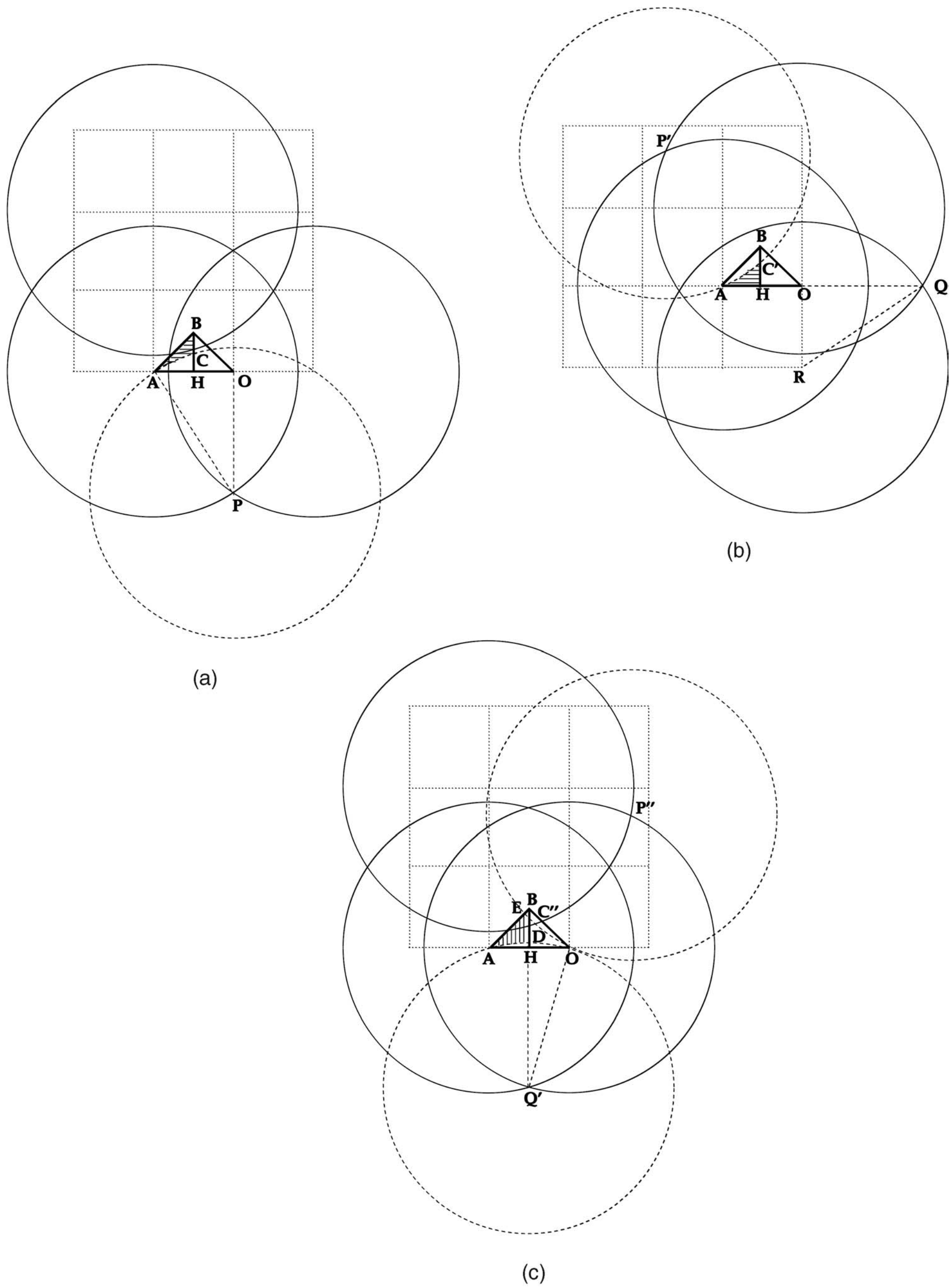


Fig. 8. CASE 5, sufficient condition. Three grid discs cover any disc centered inside the shaded areas. Intersecting the three shaded areas, we obtain triangle ABH.

covered that touch point P, given by the circle centered at point P, is defined by the equation:

$$x^2 + \left(y + \frac{\sqrt{34}}{4}\right)^2 = r^2; \tag{6}$$

this circle passes by point A and intersects segment $\overline{BH}$ at point $C = (-\frac{1}{2}, \frac{\sqrt{46}-\sqrt{34}}{4})$. It is easy to see that any disc centered inside the shaded area ABC is covered by the three grid discs centered at points: $(-1,0), (-1,2), (1,0)$. We now consider the three grid discs depicted in Fig. 8b, centered at points: $(-1,0), (0,1), (0,-1)$. First, let us focus on point Q. Its coordinates are calculated by applying the Pythagorean theorem to triangle OQR, obtaining: $Q = (\frac{\sqrt{34}}{4}, 0)$. Since $\overline{HQ} = \overline{HO} + \overline{OQ} = \frac{1}{2} + \frac{\sqrt{34}}{4} > r$, the locus of the centers of the discs to be covered that touch point Q, given by the circle centered at point Q, does not intersect triangle ABH. Now, let us focus on point P'. The coordinates of point P' are calculated by intersecting the two grid circles:

$$\begin{cases} (x+1)^2 + y^2 &= r^2 \\ x^2 + (y-1)^2 &= r^2, \end{cases} \tag{7}$$

obtaining: $P' = (\frac{-2-\sqrt{21}}{4}, \frac{2+\sqrt{21}}{4})$. Therefore, the locus of the centers of the discs to be covered that touch point P', given by the circle centered at point P', is defined by the equation:

$$\left(x + \frac{2+\sqrt{21}}{4}\right)^2 + \left(y - \frac{2+\sqrt{21}}{4}\right)^2 = r^2; \tag{8}$$

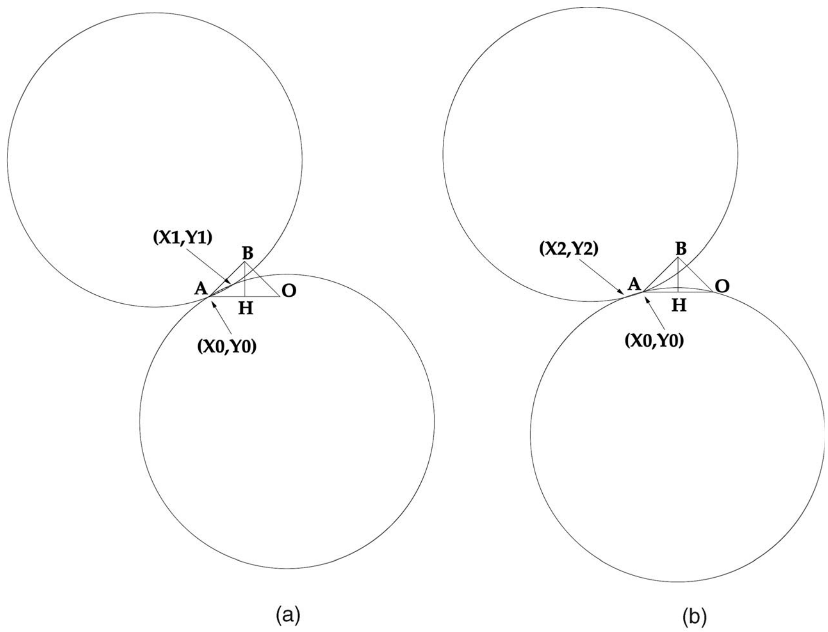


Fig. 9. CASE 5, sufficient condition. Point (x_1, y_1) is inside triangle ABH, point (x_2, y_2) is outside triangle ABH.

this circle passes by point A and intersects segment $\overline{BH}$ at point $C' = (-\frac{1}{2}, \frac{2+\sqrt{21}-\sqrt{29}}{4})$. It is easy to see that any disc centered in the shaded area $AC'H$ is covered by the three grid discs centered at points: $(-1,0),(0,1),(0,-1)$.

Intersecting the two circles centered at points P and P', defined by (6) and (8), respectively, we obtain two points:

$$(x_0, y_0) = (-1, 0)$$
$$(x_1, y_1) = \left(\frac{2-\sqrt{21}}{4}, \frac{2+\sqrt{21}-\sqrt{34}}{4}\right),$$

depicted in Fig. 9a. By the coordinates of these points, (x_1, y_1) is placed inside triangle ABH. Hence, any disc centered inside the shaded area given by the intersection of the two discs depicted in Fig. 9a is covered by neither the three grid discs depicted in the upper left section nor by the three grid discs depicted in Fig. 8b.

In order to cover such a disc, we consider the three grid discs depicted in Fig. 8c, placed at points: $(-1,0),(-1,2),(0,0)$. In this case, the coordinates of point Q' are calculated by applying the Pythagorean theorem to triangle $Q'OH$, obtaining: $Q' = (-\frac{1}{2}, -\sqrt{\frac{23}{8}})$. Therefore, the locus of the centers of the discs to be covered that touch point Q', given by the circle centered at Q', is defined by the equation:

$$\left(x+\frac{1}{2}\right)^2 + \left(y+\sqrt{\frac{23}{8}}\right)^2 = r^2; \tag{9}$$

this circle passes by points A and O and intersects segment $\overline{BH}$ at point $D = (-\frac{1}{2}, \frac{5\sqrt{2}-\sqrt{46}}{4})$. The coordinates of point P'' are calculated by intersecting the two grid circles:

$$\begin{cases} (x+1)^2 + (y-2)^2 = r^2 \\ x^2 + y^2 = r^2, \end{cases} \tag{10}$$

obtaining: $P'' = (\frac{\sqrt{6}-1}{2}, \frac{\sqrt{6}+4}{4})$. Therefore, the locus of the centers of the discs to be covered that touch point P'', given by the circle centered in P'', is defined by the equation:

$$\left(x-\frac{\sqrt{6}-1}{2}\right)^2 + \left(y-\frac{\sqrt{6}+4}{4}\right)^2 = r^2; \tag{11}$$

this circle passes by point O and intersects segment $\overline{BH}$ at point $C'' = (-\frac{1}{2}, \frac{4+\sqrt{6}-\sqrt{26}}{4})$. It is easy to see that any disc centered inside the shaded area $AEC''D$ is covered by the three grid discs centered at points: $(-1,0),(-1,2),(0,0)$. In order to check that this placement of grid discs covers any disc centered inside the shaded area given by the intersection of the two discs depicted in Fig. 9a, we intersect the two circles centered at points P' and Q' in Fig. 8, defined by (8) and (9), respectively, obtaining two points:

$$(x_0, y_0) = (-1, 0)$$
$$(x_2, y_2) = \left(-\frac{\sqrt{21}}{4}, \frac{2+\sqrt{21}-\sqrt{46}}{4}\right),$$

depicted in Fig. 9b. Given the coordinates of these points, since (x_2, y_2) is placed outside the area of triangle ABH, we conclude that the three grid discs placed at points $(-1,0),(-1,2),(0,0)$ cover any disc centered inside the shaded area given by the intersection of the two discs depicted in Fig. 9a. It follows that intersecting the three shaded areas of Fig. 8: ABC, $AC'H$, $AEC''D$, we cover triangle ABH completely. Hence, any disc centered inside triangle ABH is covered by three grid discs, placed in one of the three configurations depicted in Fig. 8. By symmetry, the same holds for triangle OBH and we can tile the plane with triangles inside which discs can be centered and covered by three grid discs.

3 Conclusions

We presented a basic theorem in combinatorial geometry that can be applied to design facility location algorithms with constant approximation factors and to determine if a grid network design is cost effective when compared to a nongrid design.

Future work includes generalization of the theorem to other lattice structures. We also plan to study a related problem also inspired by wireless communication networks. That is the problem of placing a minimum number of discs of given radius r on the plane in a way that they cover a set of given points and that the resulting graph, which has the centers of the discs as vertices and vertices joined by an edge if the corresponding discs intersect, is connected.

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